

WHEELER-DEWITT EQUATION WITH VARIABLE CONSTANTS.

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In this paper we study how all the physical “constants” vary in the framework described by a model in which we have taken into account the generalize conservation principle for its stress-energy tensor. This possibility enable us to take into account the adiabatic matter creation in order to get rid of the entropy problem. We try to generalize this situation by contemplating multi-fluid components. To validate all the obtained results we explore the possibility of considering the variation of the “constants” in the quantum cosmological scenario described by the Wheeler-DeWitt equation. For this purpose we explore the Wheeler-DeWitt equation in different contexts but from a dimensional point of view. We end by presenting the Wheeler-DeWitt equation in the case of considering all the constants varying. The quantum potential is obtained and the tunneling probability is studied.

Keywords: Quantum Cosmology. Time-varying constants.

1. Introduction.

In a recent paper ¹ we studied the behaviour of the “constants” G, c, Λ and \hbar in the framework described by a cosmological model with FRW symmetries in which we have imposed the equation of state $\rho = a\theta^4$ for the energy density. In this way a possible variation of the “constant” \hbar is reflected in the field equations. Formally, the obtained result only is valid for a determinate kind of matter, radiation predominance, but we extrapolated the result for other kind of matter, dust, strings, ultrastiff matter etc... checking that such extrapolation is right since any equations of physics as Maxwell, Schrödinger or Klein-Gordon equations remain gauge invariant if we introduce into them the “constants” varying for any kind of matter considered in the field equations.

Now we would like to validate the obtained result and to find the behaviour for \hbar as well as for the rest of the “constants” through a equation in which “functions” (“constants”) appear independently of the imposed equation of state. For this purpose we resort to the Wheeler-DeWitt (W-DW) equation in a suitable minisuperspace, since in this equation such “constants” for any kind of matter always appear. Therefore, in this paper we try to study the behaviour of these “constants” in the framework described by quantum cosmology.

The paper is organized as follows: In section 2 we review briefly the obtained result in reference ¹ about the variation of the constants and we shall introduce new

results that will show us how the constants vary when we consider mechanism of adiabatic matter creation. We shall explore the possibility of considering the variation of constants in the multi-fluid scenario, i.e. the stress-energy tensor is defined by various kinds of matter as for example radiation, dust and strings. Once these results have been exposed, in section 3 we shall study the Wheeler-DeWitt equation in the minisuperspace approximation i.e. we shall impose FRW symmetries. Obviously, all the “constants” will be taken into account when we write such equation and we shall study it from the dimensional point of view. We will obtain the reduced form of the W-DW equation i.e. the Schrödinger equation, in such a way that all the quantum procedure could be used. We shall check that the obtained results in section 2 remain gauge invariant the W-DW equation. In section 4 we will present another model described by a scalar field, also studying it from a dimensional point of view. Finally, in section 5 we shall study both equation, W-DW equation as well as the Schrödinger’s one (its reduced form) with all the constants varying. We will obtain its potential and the tunneling probability is studied. To calculate such probability we employ the Gamow’s classical formula in which all the “constants” appear, not only the potential as in previous proposals, since we believe that their variations must be reflected in such calculation.

2. Variable constants.

In a recent paper¹ we studied a flat cosmological model with FRW symmetries where the energy-momentum tensor is defined by a perfect fluid. As we have indicated in the introduction, in order to reflect the possible variation of “constant” \hbar into the field equations, we impose the equation of state $\rho = a\theta^4$ for the energy density ρ , where θ stands for the temperature and a is the radiation constant.

The field equations are:

$$2\frac{f''}{f} + \frac{(f')^2}{f^2} = -\frac{8\pi G(t)}{c(t)^2}p + c(t)^2\Lambda(t), \quad (1)$$

$$3\frac{(f')^2}{f^2} = \frac{8\pi G(t)}{c(t)^2}\rho + c(t)^2\Lambda(t). \quad (2)$$

The differential equation that describes the variation of all the “constants” is obtained from the condition

$$\text{div} \left(\frac{8\pi G}{c^4} T_i^j + \delta_i^j \Lambda \right) = 0, \quad (3)$$

that simplified is:

$$T_{i;j}^j - \left(\frac{4c_{,j}}{c} - \frac{G_{,j}}{G} \right) T_i^j + \frac{c^4(t)\delta_i^j \Lambda_{,j}}{8\pi G} = 0 \quad (4)$$

or equivalently:

$$4\frac{\theta'}{\theta} - 3 \left[\frac{c'}{c} + \frac{\hbar'}{\hbar} \right] + 3(\omega + 1)H + \frac{15\Lambda' c^7 \hbar^3}{8\pi^3 G k_B^4 \theta^4} + \frac{G'}{G} - 4\frac{c'}{c} = 0. \quad (5)$$

To solve this equation we imposed two simplifying hypotheses, the first one, that the relation $G/c^2 = B$ (where B is a *const.*) remains constant, and the second one, that the cosmological “constant” verifies the relation $\Lambda \propto \frac{d}{c^2 t^2}$ with $d \in \mathbb{R}$, while

$\text{div} (T_i^j) \neq 0$, in this way the equation was solved perfectly. Since the constant B has dimensions $[B] = LM^{-1}$ we can get the dimensionless monomia $\pi_1 = \frac{\rho B t^2}{b}$ where $b \in \mathbb{R}$. With these hypotheses and π_1 equation (5) simplifies to:

$$4 \frac{\theta'}{\theta} - 3 \left[\frac{c'}{c} + \frac{\hbar'}{\hbar} \right] + 3(\omega + 1)H - \frac{15d}{4\pi^3} \frac{c^4 [c't + c] \hbar^3}{G k_B^4 \theta^4 t^3} + \frac{G'}{G} - 4 \frac{c'}{c} = 0, \quad (6)$$

that has no immediate integration. We have to take into account the field equations (2)

$$3H^2 = 8\pi b t^{-2} + d t^{-2}, \quad (7)$$

from this we get $f = K_{\varkappa} t^{\varkappa}$ where $\varkappa = \left(\frac{8\pi b + d}{3} \right)^{\frac{1}{2}}$ and substituting in (6) together with $G = Bc^2$ we get

$$4 \frac{\theta'}{\theta} - 3 \left[\frac{c'}{c} + \frac{\hbar'}{\hbar} \right] + \frac{3(\omega + 1)\varkappa}{t} - \frac{15d}{4\pi^3} \frac{c^2 [c't + c] \hbar^3}{B k_B^4 \theta^4 t^3} - 2 \frac{c'}{c} = 0, \quad (8)$$

i.e. one equation with 3 unknowns.

If we want to integrate equation (8) we have to take a decision on the behavior of the constant \hbar .

Taking $c\hbar = \text{const.} \approx \hbar = \frac{A}{c}$ then $\frac{\hbar'}{\hbar} = -\frac{c'}{c}$ yielding:

$$4 \frac{\theta'}{\theta} + \frac{3(\omega + 1)\varkappa}{t} - \frac{15d}{4\pi^3} \frac{A^3 [c't + c]}{c B k_B^4 \theta^4 t^3} - 2 \frac{c'}{c} = 0. \quad (9)$$

Also if $\rho = a\theta^4$ and $\rho = \frac{b}{Bt^2} \implies \frac{b}{Bt^2} = a\theta^4$ we have:

$$k_B \theta = \left(\frac{15c^3(t)\hbar^3(t)b}{\pi^2 B t^2} \right)^{\frac{1}{4}} = \left(\frac{15A^3 b}{\pi^2 B} \right)^{\frac{1}{4}} t^{-1/2}. \quad (10)$$

And substituting into the previous equation we get :

$$\frac{-2}{t} + \frac{3(\omega + 1)\varkappa}{t} - \frac{15d}{4\pi^3} \frac{A^3 [c't + c]}{c B t^3 \frac{15A^3 b}{\pi^2 B t^2}} - 2 \frac{c'}{c} = 0, \quad (11)$$

and simplifying

$$\frac{-2}{t} + \frac{3(\omega + 1)\varkappa}{t} - \frac{d}{4\pi b} \left[\frac{c'}{c} + \frac{1}{t} \right] - 2 \frac{c'}{c} = 0, \quad (12)$$

therefore we get a very simple differential equation:

$$\frac{c'}{c} = \left[\frac{12\pi b(\omega + 1)\varkappa - 8\pi b - d}{8\pi b + d} \right] \frac{1}{t} \quad (13)$$

integrating it we obtain easily:

$$c = K_{\xi} t^{\xi} \quad (14)$$

where $\xi = \left[\frac{12\pi b(\omega + 1)\varkappa - 8\pi b - d}{8\pi b + d} \right]$.

We can consider another possibility. Take the group of governing quantities $\mathfrak{M} = \mathfrak{M}\{K_{\varkappa}, A, t\}$ where K_{\varkappa} is the proportionality constant obtained from $f = K_{\varkappa}t^{\varkappa}$ and A is the constant establishing the relation between \hbar and c . The results obtained by means of the gauge relations are:

$$\begin{aligned} G &\propto K_{\varkappa}^6 A^{-1} t^{6\varkappa-4} \\ c &\propto K_{\varkappa} t^{\varkappa-1} \\ \hbar &\propto K_{\varkappa}^{-1} A t^{1-\varkappa} \\ k_B \theta &\propto K_{\varkappa}^{-1} A t^{-\varkappa} \\ \rho &\propto K_{\varkappa}^{-4} A t^{-4\varkappa} \\ m_i &\propto K_{\varkappa}^3 A t^{-3\varkappa+2} \\ \Lambda &\propto K_{\varkappa}^{-2} t^{-2\varkappa} \\ e^2 \varepsilon_0^{-1} &\propto A \end{aligned} \quad (15)$$

where m_i comes from the energy density definition $\rho_E = \frac{nm_i c^2}{f^3}$ (n stands for the particles number) and $e^2 \varepsilon_0^{-1}$ from the definition of the fine structure constant α . We can check that we recover the general covariance property $\frac{G}{c^2} = \text{const.}$ iff $\varkappa = \frac{1}{2}$. Similarly we can see that the following relations are satisfied: $\rho = a\theta^4$, $\rho = A f^{-4}$ (equivalent to $\text{div}(T) = 0$), $\Lambda \propto f^{-2}$ and $f = ct$ (no horizon problem). And finally $e^2 \varepsilon_0^{-1} \propto \text{const.}$ In this way the fine structure constant $\alpha \propto \frac{e^2}{\varepsilon_0 c \hbar} = \text{const.}$, is a true constant. With the value $\varkappa = \frac{1}{2}$ we get

$$\begin{aligned} c &\propto t^{-1/2}, & \hbar &\propto t^{1/2}, & G &\propto t^{-1}, & k_B \theta &\propto t^{-1/2} \\ f &\propto t^{1/2}, & \rho &\propto t^{-2}, & m_i &\propto t^{1/2} \end{aligned} \quad (16)$$

Another alternative is to consider the governing quantities $\mathfrak{M} = \mathfrak{M}\{B, A, t\}$ such that $[A] = [A_\omega]$ with $\omega = \frac{1}{3}$. So that we get the same problem as with the condition $\text{div}(T) = 0$ (Note $\rho = A_\omega f^{-3(\omega+1)}$) This would imply no previous hypothesis on the behavior of \hbar . With these results we have seen that the equations of the physic like Maxwell, Schrödinger or Klein-Gordon equations remain gauge invariant. We emphasize that in such work we determined the behaviour of \hbar without any doubt within this framework (see¹ for details).

In this paper we would like to study briefly the important case in which the condition $\text{div}(T_i^j) = 0$ is taken into account in such a way that adiabatic matter creation can be taken into account, in order to get rid of the entropy problem since in¹ this problem was no solved.

With these new assumptions the field equations that now govern the model are as follows:

$$2\frac{f''}{f} + \frac{(f')^2}{f^2} = -\frac{8\pi G(t)}{c^2(t)}(p + p_c) + c^2(t)\Lambda(t), \quad (17)$$

$$3\frac{(f')^2}{f^2} = \frac{8\pi G(t)}{c^2(t)}\rho + c^2(t)\Lambda(t), \quad (18)$$

$$n' + 3nH = \psi, \quad (19)$$

and taking into account our general assumption i.e.

$$T_{i;j}^j - \left(\frac{4c_{,j}}{c} - \frac{G_{,j}}{G}\right)T_i^j + \frac{c^4(t)\delta_i^j \Lambda_{,j}}{8\pi G} = 0 \quad (20)$$

with $T_{i;j}^j = 0$, this brings us to obtain two equations

$$\rho' + 3(\rho + p + p_c)H = 0 \quad (21)$$

and

$$\frac{\Lambda' c^4}{8\pi G \rho} + \frac{G'}{G} - 4\frac{c'}{c} = 0 \quad (22)$$

where n measures the particles number density, ψ is the function that measures the matter creation, $H = f'/f$ represents the Hubble parameter (f is the scale factor that appears in the metric), p is the thermostatic pressure, ρ is energy density and p_c is the pressure that generates the matter creation.

The creation pressure p_c depends on the function ψ . For adiabatic matter creation this pressure takes the following form:

$$p_c = - \left[\frac{\rho + p}{3nH} \psi \right]. \quad (23)$$

The state equation that we next use is the well-known expression

$$p = \omega \rho \quad (24)$$

where $\omega = \text{const.}$ $\omega \in (-1, 1]$. We assume that this function follows the law:

$$\psi = 3\beta nH, \quad (25)$$

(see ²⁾ where β is a dimensionless constant (if $\beta = 0$ then there is no matter creation since $\psi = 0$). The generalized principle of conservation $T_{i;j}^j = 0$, for the stress-energy tensor (21) brings us to:

$$\rho' + 3(\omega + 1)\rho \frac{f'}{f} = (\omega + 1)\rho \frac{\psi}{n}. \quad (26)$$

By integrating equation (26) we obtain the following relation between the energy density and the scale factor and which is more important, the constant of integration that we shall need for our subsequent calculations:

$$\rho = A_{\omega,\beta} f^{-3(\omega+1)(1-\beta)}, \quad (27)$$

where $A_{\omega,\beta}$ is the integration constant that depends on the equation of state that we want to consider i.e. constant ω and constant β that controls the matter creation, $[A_{\omega,\beta}] = L^{3(\omega+1)(1-\beta)-1} M T^{-2}$. With this constant of integration and taking into account the hypothesis about the relation $G/c^2 = B$, our purpose is to show that no more hypothesis are necessary to solve the differential equations that govern the model. Therefore the set of governing parameters are now: $\mathfrak{M} = \mathfrak{M}\{A_{\omega,\beta}, B, t\}$,

that brings us to obtain the next relations:

$$\begin{aligned}
G &\propto A_{\omega,\beta}^{\frac{2}{\gamma+1}} B^{\frac{2}{\gamma+1}+1} t^{\frac{2(1-\gamma)}{\gamma+1}}, \\
c &\propto A_{\omega,\beta}^{\frac{1}{\gamma+1}} B^{\frac{1}{\gamma+1}} t^{\frac{(1-\gamma)}{\gamma+1}}, \\
\hbar &\propto A_{\omega,\beta}^{\frac{3}{\gamma+1}} B^{\frac{3}{\gamma+1}-1} t^{\frac{5-\gamma}{\gamma+1}}, \\
m_i &\propto A_{\omega,\beta}^{\frac{1}{\gamma+1}} B^{-\frac{\gamma}{\gamma+1}} t^{\frac{2}{\gamma+1}}, \\
e^2 \varepsilon_0^{-1} &\propto A_{\omega,\beta}^{\frac{\gamma-3}{\gamma+1}+1} B^{\frac{\gamma-3}{\gamma+1}} t^{2\frac{\gamma-3}{\gamma+1}}, \\
\rho &\propto B^{-1} t^{-2}, \\
f &\propto A_{\omega,\beta}^{\frac{1}{\gamma+1}} B^{\frac{1}{\gamma+1}} t^{\frac{2}{\gamma+1}}, \\
k_B \theta &\propto A_{\omega,\beta}^{\frac{3}{\gamma+1}} B^{\frac{3}{\gamma+1}-1} t^{\frac{4-2\gamma}{\gamma+1}}, \\
a^{-1/4} s &\propto A_{\omega,\beta}^{\frac{3}{\gamma+1}} B^{\frac{3}{\gamma+1}-\frac{3}{4}} t^{\frac{6}{\gamma+1}-\frac{3}{2}}, \\
\Lambda &\propto A_{\omega,\beta}^{\frac{-2}{\gamma+1}} B^{\frac{-2}{\gamma+1}} t^{\frac{-4}{\gamma+1}}, \\
q &= \frac{\gamma-1}{2}
\end{aligned} \tag{28}$$

where $\gamma = 3(\omega + 1)(1 - \beta) - 1$, and q is the deceleration parameter. If $\omega = 1/3$ then it is observed that the entropy is not constant

$$a^{-1/4} s \propto t^{\frac{3}{2(1-\beta)} - \frac{3}{2}}, \tag{29}$$

we can check that the next results are verified: we see that $\frac{G}{c^2} = B \forall \beta$ (trivially), $\rho = a\theta^4 \forall \beta$, $f = ct \forall \beta$, $\Lambda \propto \frac{1}{c^2 t^2} \propto f^{-2}$ while the relation $\hbar c \neq \text{const.}$ since it depends on β . We also can check that our model has no the so called Planck's problem since the Planck system behaves now as:

$$\begin{aligned}
l_p &= \left(\frac{G\hbar}{c^3}\right)^{1/2} \approx f(t), \\
m_p &= \left(\frac{c\hbar}{G}\right)^{1/2} \approx f(t), \\
t_p &= \left(\frac{G\hbar}{c^5}\right)^{1/2} \approx t,
\end{aligned} \tag{30}$$

since the radius of the Universe $f(t)$ at Planck's epoch coincides with the Planck's length $f(t_p) \approx l_p$, while the energy density at Planck's epoch coincides with the Planck's energy density $\rho(t_p) \approx \rho_p \approx t^{-2}$, where $\rho_p = m_p c^2 / l_p^3$. See² for more details and the followed method etc...

It is observed from (28) that if we make $\beta = 0$ the following set of solutions are

obtained:

ω	1	2/3	1/3	0	-1/3	-2/3
f	1/3	2/5	1/2	2/3	1	2
ρ	-2	-2	-2	-2	-2	-2
θ	-1	-4/5	-1/2	0	1	4
s	-1/2	-3/10	0	1/2	3/2	9/2
G	-4/3	-6/5	-1	-2/3	0	2
c	-2/3	-3/5	-1/2	-1/3	0	1
\hbar	0	1/5	1/2	1	2	5
m_i	1/3	2/5	1/2	2/3	1	2
Λ	-2/3	-4/5	-1	-4/3	-2	-4
$e^2 \varepsilon_0^{-2}$	-2/3	-2/5	0	2/3	2	6
q	5/2	3/2	1	1/2	0	-1/2

(31)

with: ($\omega = -1$) corresponds to de Sitter (false vacuum) represented by the cosmological constant (special case), ($\omega = -\frac{2}{3}$) for domain walls, ($\omega = -\frac{1}{3}$) for strings, ($\omega = 0$) for dust (matter predominance), ($\omega = \frac{1}{3}$) for radiation or ultrarelativistic gases (radiation predominance), ($\omega = \frac{2}{3}$) for perfect gases, ($\omega = 1$) for ultra-stiff matter. For example, if we take the case $\omega = 0$ the table (31) tells us that

$$f \propto t^{2/3}, \quad \rho \propto t^{-2}, \quad \theta \propto t^0 = \text{const.}, \dots$$

$$G \propto t^{-2/3}, \quad c \propto t^{-1/3}, \quad \hbar \propto t, \dots \text{etc.}$$

This table tells us too that if we want that our universe accelerates then we have to impose that $-1 \leq \omega < -1/3$ but we must be careful since with this parameter we see that the temperature increases.

We can try to generalize this scenario taking into account various kinds of matter. The idea is as follow. We can define a general energy density $\tilde{\rho}$ as:

$$\tilde{\rho} = \sum_{i=0}^6 \rho_i \quad (32)$$

where ρ_i stands for each kind of energy density, and the parameter $i = 0, 1, \dots, 6$ in such a way that: $i = 0$ correspond to $\omega = -1$ (the false vacuum), $i = 1$ correspond to domain walls i.e. to $\omega = -2/3$, $i = 2$ to $\omega = -1/3$, $i = 3$ to $\omega = 0$, $i = 4$ to $\omega = 1/3$, $i = 5$ to $\omega = 2/3$ and finally $i = 6$ to $\omega = 1$; but in such a way that each type of matter verifies the relation

$$p_i = \omega_i \rho_i \quad (33)$$

in this way we define the total pressure as:

$$\tilde{p} = \sum_{i=0}^6 p_i \quad (34)$$

but in this case we do not impose that it is verified for each kind of matter the relation:

$$\rho_i = A_{\omega_i} f^{-3(\omega+1)} \quad (35)$$

Our purpose is as follows: we impose that the relation $\text{div}(\tilde{T}) = 0$ it is verified i.e.

$$\tilde{\rho}' + 3(\tilde{p} + \tilde{\rho})H = 0 \quad (36)$$

taken into account that $p_i = \omega_i \rho_i$ then

$$\tilde{\rho}' + 3H \sum_{i=0}^6 [(\omega_i + 1) \rho_i] = 0 \quad (37)$$

that we can “approximate” through the relation:

$$\tilde{\rho} = A_m f^{-m} \quad (38)$$

with $m = 3 \sum_{i=0}^6 (\omega_i + 1)$. It is proven in a trivial way that if we consider only one type of matter we then recuperate the above results i.e.. $m = 3(\omega + 1)$.

Therefore with the next set of governing quantities $\mathfrak{M} = \mathfrak{M}(A_m, B, t)$ we arrive to obtain the following table of results:

$$\begin{aligned} G &\propto A_m^{\frac{2}{x+1}} B^{\frac{2}{x+1}} t^{\frac{4}{x+1}-2} \\ c &\propto A_m^{\frac{1}{x+1}} B^{\frac{1}{x+1}} t^{\frac{2}{x+1}-1} \\ \Lambda &\propto A_m^{\frac{-2}{x+1}} B^{\frac{-2}{x+1}} t^{\frac{-4}{x+1}} \\ \hbar &\propto A_m^{\frac{3}{x+1}} B^{\frac{3}{x+1}-1} t^{\frac{6}{x+1}-1} \\ m_i &\propto A_m^{\frac{1}{x+1}} B^{\frac{-x}{x+1}} t^{\frac{2}{x+1}} \\ e^2 \varepsilon_0^{-2} &\propto A_m^{\frac{4}{x+1}} B^{\frac{4}{x+1}-1} t^{\frac{8}{x+1}-2} \\ f &\propto A_m^{\frac{1}{x+1}} B^{\frac{1}{x+1}} t^{\frac{2}{x+1}} \\ \rho &\propto B^{-1} t^{-2} \\ k_B \theta &\propto A_m^{\frac{1}{x+1}} B^{\frac{3}{x+1}-1} t^{\frac{6}{x+1}-2} \\ a^{-1/4} s &\propto A_m^{\frac{3}{x+1}} B^{\frac{3}{x+1}-\frac{3}{4}} t^{\frac{6}{x+1}-\frac{3}{2}} \\ q &= \frac{x+1}{2} - 1 \end{aligned} \quad (39)$$

where $x + 1 = m = 3 \sum_{i=0}^6 (\omega_i + 1)$. If for example we consider an universe with dust ($\omega = 0$) and radiation ($\omega = 1/3$) then $m = 9$, we obtain a very surprising results as we can see

$$G \propto t^{-8/5}, \quad c \propto t^{-4/5}, \quad \hbar \propto t^{-2/5}, \quad \Lambda \propto t^{-2/5}, \quad \rho \propto t^{-2}, \quad f \propto t^{1/5} \quad \text{etc...} \quad (40)$$

We have developed this section since when we attack in the next section the Wheeler-DeWitt equation we shall see that in it we can take into account all these kinds of matter, for this reason we need to know beforehand how all the constants in these frameworks vary. .

3. Wheeler-DeWitt equation. Minisuperspace approximation.

3.1. Case 1. General case.

The Einstein equation has precisely the same form as the Hamiltonian for a zero-energy particle whose position is described by a coordinate f . To quantize

we must replace the momentum conjugate to f by its corresponding operator, according to:

$$p_f \rightarrow \hat{p}_f = -i\hbar\partial_f \quad (41)$$

The Einstein equations may be obtained via Hamilton's principle³. From the Einstein field equations we define our Lagrangian as:

$$L = -\aleph f^3 \left[\left(\frac{f'}{f} \right)^2 - \frac{kc^2}{f^2} + \frac{8\pi G}{3c^2}(\tilde{\rho}) \right] \quad (42)$$

where $\aleph = \frac{3\pi c^2}{4G}$ is a renormalization factor and $\tilde{\rho}$ stands for all the possible forms of energy density. In the case of FRW universe the action takes the form:

$$S_{grav} = \int L_{grav} dt = \aleph \int f^3 \left[- \left(\frac{f'}{f} \right)^2 + \frac{kc^2}{f^2} - \frac{8\pi G}{3c^2}(\tilde{\rho}) \right] dt \quad (43)$$

The momentum conjugate to f is:

$$p \equiv \frac{\partial L}{\partial f'} = -2\aleph f f' \quad (44)$$

We define the Hamiltonian $H(f', f)$ as:

$$H = p f' - L \quad (45)$$

simplifying it yields

$$H = -\aleph f^3 \left[\left(\frac{f'}{f} \right)^2 + \frac{kc^2}{f^2} - \frac{8\pi G}{3c^2}(\tilde{\rho}) \right] \quad (46)$$

the Hamiltonian has been written in terms of f' to show explicitly that it is identically zero and is not equal to the total energy $H = 0$. In terms of the conjugate momentum p , the Einstein equations may be written as follow:

$$H(p, f) = -\aleph f^3 \left[\frac{p^2}{4\aleph^2 f^4} + \frac{kc^2}{f^2} - \frac{8\pi G}{3c^2}(\tilde{\rho}) \right] = 0 \quad (47)$$

which, of course is also equal to zero. Straightforward algebra it yields:

$$p^2 + 4\aleph^2 f^2 \left(kc^2 - \frac{8\pi G}{3c^2} f^2(\tilde{\rho}) \right) = 0 \quad (48)$$

Quantizing, making the replacement

$$\hat{p} \longrightarrow -i\hbar\partial_f \quad (49)$$

and imposing $\hat{H}\psi = 0$ results the Wheeler-DeWitt equation in the minisuperspace approximation for arbitrary k and with different kinds of matter expressed in $\tilde{\rho}$.

Now, if we take into account that $\aleph = \frac{3\pi c^2}{4G}$ then

$$\left[\frac{d^2}{df^2} - \frac{9\pi^2 c^4 f^2}{4\hbar^2 G^2} \left(kc^2 - \frac{8\pi G}{3c^2}(\tilde{\rho}) f^2 \right) \right] \psi = 0 \quad (50)$$

and if we replace $\tilde{\rho}$ by (for example) $\tilde{\rho} = \rho + \rho_{vac.}$ where ρ stands for energy density of matter or radiation and $\rho_{vac.}$ is the energy density of the vacuum expressed in terms of the cosmological constant

$$\rho_{vac.} = \frac{\Lambda c^4}{8\pi G} \quad (51)$$

Note that other types of matter can be taken into account like for example strings, ultrastiff matter, domains walls etc... algebra brings us to the following expression

$$\left[\frac{d^2}{df^2} - \frac{9\pi^2 c^6 f^2}{4\hbar^2 G^2} \left(k - \frac{\Lambda f^2}{3} - \frac{8\pi G}{3c^4} A_\omega f^{-1-3\omega} \right) \right] \psi = 0 \quad (52)$$

where we have used the relation between ρ and f

$$\rho = A_\omega f^{-3(\omega+1)} \quad (53)$$

or in a compact notation

$$\frac{d^2\psi}{df^2} - V(f)\psi = 0 \quad (54)$$

where

$$V(f) = \frac{9\pi^2 f^2}{4l_p^4} \left(k - \frac{\Lambda f^2}{3} - \frac{8\pi G}{3c^4} A_\omega f^{-1-3\omega} \right) \quad (55)$$

being l_p Planck's length.

The W-DW eq. is identical to one-dimensional time-independent Schrödinger eq. for a one-half unit mass particle of zero energy subject to the potential

$$V(f) = \frac{9\pi^2}{4l_p^4} f^2 \left(k - \frac{\Lambda f^2}{3} - \frac{8\pi G}{3c^4} A_\omega f^{-1-3\omega} \right) \quad (56)$$

The “particle” at $f = 0$ - a quantum FRW universe of zero radius or indeed our cosmological “nothing”- may quantum mechanically tunnel through the potential barrier to appear at $f = f_0$. This tunneling event represents a FRW universe of size (scale factor) f_0 that has quantum mechanically popped into existence i.e. a universe that has been created spontaneously and nonsingularly (since $f_0 \neq 0$). Choosing the tunneling wave function, it is now a simple matter to calculate the probability with which this occurs. If we denote the amplitude for the quantum creation by

$$|\langle FRW(f_0) | nothing \rangle|^2 = P \quad (57)$$

$$P \simeq \exp \left[-\frac{2}{\hbar} \int_0^{f_0} \sqrt{2m_p [V(f')] } df' \right] \quad (58)$$

The tunneling probability of the universe will be obtained from the expression above (see ^{4,5}).

3.2. The dimensional stone.

We now calculate the multiplicity of the dimensional base ⁶. For this purpose we observe that equation (52)

$$\frac{\partial^2 \psi}{\partial f^2} - \frac{9\pi^2 c^4 f^2}{4\hbar^2 G^2} \left(kc^2 - \frac{\Lambda c^2 f^2}{3} - \frac{8\pi G}{3c^2} A_\omega f^{-1-3\omega} \right) \psi = 0 \quad (59)$$

can be written in the following dimensionless products as:

$$\pi_1 := \frac{c^6 f^4}{\hbar^2 G^2} \quad \pi_2 := \frac{c^6 f^6 \Lambda}{\hbar^2 G^2} \quad \pi_3 := \frac{c^2 A_\omega f^{3(1-\omega)}}{\hbar^2 G} \quad (60)$$

from π_1 it is observed that $l_p \propto f$, from π_2 and π_1 we can see that $\Lambda \propto f^{-2}$ and from π_3 it is observed that $\hbar \propto A_\omega^{1/2} B^{-1/2} f^{3(1-\omega)/2}$. We shall see in section 4 that these results are precisely what we obtain when the possibility of time-varying constants in GR is considered in section 2 but with $\beta = 0$. We would like to emphasize that while in these cases it was studied the variation of the “constants” in semiclassical cosmological models where constant \hbar was introduced through an equation of state and to extrapolate the result to any kind of matter the result obtained here does not depend on any equation of state. It is observed that it only depends on the kind of matter, this fact is reflected in constant A_ω . In this way we validate our extrapolation as well as the obtained results in section 2. With these relations (60) we can say that the Wheeler-DeWitt equation remains gauge invariant, see ⁽¹⁾

We proceed to calculate the multiplicity of the base in this model. The rank of the matrix of the exponents of the quantities and constants included in the monomia is 3 as it results immediately from:

	G	c	\hbar	Λ	A_ω	f
π_1	-2	6	-2	0	0	4
π_2	-2	6	-2	1	0	6
π_3	-1	2	-2	0	1	$3(1-\omega)$

(61)

The multiplicity of the dimensional base is therefore $m = (\text{number of quantities and constants}) - (\text{rank of the matrix})$ is 3. Thus we can use as base $B = \{G, c, \hbar\}$. The only fundamental quantity is f and the set of unavoidable constants are $C = \{\Lambda, A_\omega, G, c, \hbar\}$. It is observed that the set of fundamental constants $[G, c, \hbar]$ form the famous Planck’s system of units and this possibility should be taken into account.

Once the dimensional base of the theory (model) is obtained we go next to calculate the dimensional equation of each quantity, these are:

$$[G] = G \quad [c] = c \quad [\hbar] = \hbar \quad [k_B] = k_B \quad (62)$$

since we can take into account the equation of state $\rho = a\theta^4$ where a stands for the radiation constant ($a \propto k_B^4/\hbar^3 c^3$). The dimensional equations of the rest of the quantities are:

$$\begin{aligned} [f] &= \hbar^{1/2} c^{-3/2} G^{1/2} & [\rho] &= \hbar^{-1} c^7 G^{-2} \\ [A_0] &= \hbar^{1/2} c^{1/2} G^{-1/2} & [t] &= \hbar^{1/2} c^{-5/2} G^{1/2} \\ [\theta] &= \hbar^{1/2} c^{5/2} G^{-1/2} k_B^{-1} & [V(f)] &= \hbar^{-1} c^3 G^{-1} \end{aligned} \quad (63)$$

$$[\psi] = \hbar^{-n/4} c^{3n/4} G^{-n/4} \quad \text{from} \quad \int \psi \psi^* * d\Omega = 1 \quad (64)$$

where $*d\Omega$ is the volume-element on superspace, $*$ being the Hodge dual in the supermetric. In this case it is:

$$[\psi] = \hbar^{-1/4} c^{3/4} G^{-1/4} \quad (65)$$

for more details about the dimensions of the wave function see ⁽⁷⁾

It is trivially observed that:

$$\begin{aligned} [f] &= l_p & [\rho] &= \rho_p \\ [A_0] &= m_p & [t] &= t_p \\ [\theta] &= \theta_p & & \text{etc...} \end{aligned} \quad (66)$$

i.e. the dimensional equation of f i.e. $[f]$ is precisely the definition of the length of Planck etc... and which is important, we obtain the equivalence between $B = \{G, c, \hbar, k_B\}$ and $B = \{L, M, T, \theta\}$ i.e. between the obtained base from the Wheeler-DeWitt equation (derived from Einstein's equations) and the obtained one from the traditional Friedmann's equations. It is observed that the Planck system of units acquires full sense within this framework.

3.3. Schrödinger reduction.

Note that the Wheeler-DeWitt equation is defined on curved "space-time" but it is possible to reduce it to the Schrödinger one in an effective flat space which permits the conventional quantum-mechanical procedure to be used (for example the calculation of the penetration factor). Therefore the Schrödinger equation after straightforward algebra yields⁵:

$$\frac{\hbar^2}{2m_p} \frac{d^2\psi}{df^2} - \frac{9\pi^2}{8} m_p c^2 \frac{f^2}{l_p^2} \left(k - \frac{\Lambda f^2}{3} - \frac{8\pi B A_\omega}{3c^2} f^{-1-3\omega} \right) \psi = 0, \quad (67)$$

where B is $B = G/c^2$ and the potential, in this case, is defined by:

$$U(f) = \frac{9\pi^2}{8} m_p c^2 \frac{f^2}{l_p^2} \left(k - \frac{\Lambda f^2}{3} - \frac{8\pi B A_\omega}{3c^2} f^{-1-3\omega} \right), \quad (68)$$

it is observed that now this potential has dimensions of energy. Writing it in a dimensionless form we obtain the following π -monomia.

$$\tilde{\pi}_1 = \frac{m_p^2 c^2 f^4}{\hbar^2 l_p^2} \quad \tilde{\pi}_2 = \frac{m_p^2 c^2 f^6 \Lambda}{\hbar^2 l_p^2} \quad \tilde{\pi}_3 = \frac{m_p^2 B A_\omega f^{3(1-\omega)}}{\hbar^2 l_p^2} \quad (69)$$

simplifying them it is observed that they are the same monomia that the obtained ones in equation (60). Therefore this new equation has the same dimensional base that the obtained one in the case of the Wheeler-DeWitt equation. We shall use equation (67) in all our calculations.

3.4. Case 2. Scalar Field Model.

The density and pressure for an interacting scalar field ϕ is:

$$\rho_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi) \quad (70)$$

and

$$p_\phi = \frac{1}{2} \dot{\phi}^2 - V(\phi) \quad (71)$$

where $V(\phi)$ is the interaction potential. Assuming that the scalar field dominates, the field equation becomes

$$H^2 \equiv \left(\frac{f'}{f} \right)^2 = \frac{8\pi G}{3c^2} \left[\frac{1}{2} \dot{\phi}^2 + V(\phi) \right] \quad (72)$$

and instead of the ordinary conservation principle, one obtains (assuming a massless field and ignoring spatial derivatives)

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0 \quad (73)$$

which is the equation of motion for the scalar field where $V' \equiv \frac{dV}{d\phi}$.

The usual way of quantizing the FRW model for a scalar field is to treat ϕ and f as independent variables each with their own canonical momenta.

We define our Lagrangian as:

$$L = 2\pi^2 \left\{ \left(-\frac{3c^2}{8\pi G} \right) \left[f (f')^2 - f c^2 K \right] + \frac{1}{2} f^3 \dot{\phi}^2 - f^3 V(\phi) \right\} \quad (74)$$

from the field equation:

$$\left(\frac{f'}{f} \right)^2 = \frac{8\pi G}{3c^2} \left[\frac{1}{2} \dot{\phi}^2 + V(\phi) \right] \quad (75)$$

the standard procedure brings us to the following expressions:

$$\frac{\partial L}{\partial f'} = \frac{-3\pi c^2 f f'}{2G} := \pi_f \quad (76)$$

$$\frac{\partial L}{\partial \dot{\phi}} = 2\pi^2 f^3 \dot{\phi} := \pi_\phi \quad (77)$$

we define the Hamiltonian as:

$$H = \pi_f f' + \pi_\phi \dot{\phi} - L \quad (78)$$

after some simplifications straightforward algebra it yields:

$$H = -\frac{G\pi_f^2}{3\pi c^2 f} + \frac{\pi_\phi^2}{4\pi f^3} + 2\pi^2 f^3 V(\phi) - \frac{3\pi c^4 f K}{4G} \quad (79)$$

$$H = -\pi_f^2 + \frac{3c^2}{4\pi G} \frac{\pi_\phi^2}{f^2} + \frac{6\pi c^2}{G} f^4 V(\phi) - \frac{9\pi^2 c^6 f^2 K}{4G^2} \quad (80)$$

Canonical quantization it yields if:

$$\hat{\pi}_f \mapsto -i\hbar \partial_f \quad (81)$$

$$\hat{\pi}_\phi \mapsto -i\hbar \partial_\phi \quad (82)$$

therefore the Wheeler-DeWitt equation in this case becomes:

$$\left[-\frac{\partial^2}{\partial f^2} + \frac{3c^2}{4\pi G} \frac{1}{f^2} \frac{\partial^2}{\partial \phi^2} + U(f, \phi) \right] \Psi = 0 \quad (83)$$

where the potential is

$$U(f, \phi) = \frac{9\pi^2 c^6}{4\hbar^2 G^2} f^2 \left[k - \frac{8\pi G}{3c^4} V(\phi) f^2 \right] \quad (84)$$

3.5. The dimensional stone.

We now calculate the multiplicity of the dimensional base. For this purpose we observe that equation (83) can be written in the following dimensionless products as:

$$\pi_1 := \frac{c^2}{G\phi^2} \quad \pi_2 := \frac{c^6 f^4}{\hbar^2 G^2} \quad \pi_3 := \frac{c^2 V(\phi) f^6}{\hbar^2 G} \quad (85)$$

We proceed to calculate the multiplicity of the base in this model. The rank of the matrix of the exponents of the quantities and constants included in the monomia is 3 as it results immediately from:

	G	c	\hbar	ϕ	V	f
π_1	-1	2	0	-2	0	0
π_2	-2	6	-2	0	0	4
π_3	-1	2	-2	0	1	6

(86)

The multiplicity of the dimensional base is therefore $m = (\text{number of quantities and constants}) - (\text{rank of the matrix})$ in this case is 3. Thus we can use as base, for example $B = \{G, c, \hbar\}$. The fundamental quantities are $[f, \phi]$ and the set of unavoidable constants are $\mathcal{C} = \{G, c, \hbar\}$, it is observed that the set of fundamental constants $[G, c, \hbar]$ form the famous Planck's system of units.

The dimensional equations of the rest of the quantities are:

$$[\phi] = cG^{-1/2} \quad [V] = \hbar^{-1} c^7 G^{-2} \quad (87)$$

4. Variable constants.

Following the results obtained in section 2 (with $\beta = 0$) (see ¹) we determine the behaviour of the constants in function on f since this quantity is the only fundamental quantity in the theory of quantum cosmology (in our approximation in this minisuperspace). As we mentioned previously, this behaviour depends on the equation of state. A simple algebra exercise brings us to obtain the next solutions where now the set of governing parameters is: $\mathfrak{M} = \mathfrak{M}\{A_\omega, B, f\}$

$$\begin{aligned} G &\propto A_\omega B^2 f^{1-x} \\ c &\propto A_\omega^{1/2} B^{1/2} f^{(1-x)/2} \\ \hbar &\propto A_\omega^{1/2} B^{-1/2} f^{(5-x)/2} \\ m_i &\propto B^{-1} f \\ \Lambda &\propto f^{-2} \end{aligned} \quad (88)$$

where $x = 3\omega + 2$ in the usual notation. It is interesting to emphasize the next relations obtained from eq. (88):

$$l_p \approx f \quad (89)$$

and that $\Lambda \propto f^{-2}$ ($\Lambda = d_0 f^{-2}$, $d_0 \in \mathbb{R}$) and $m_i \propto f$ in all cases i.e. these results do not depend on the equation of state. As we mentioned earlier all these relationships are the same as the ones obtained in equation (60)

For example, if we impose $\omega = 1/3$ it is obtained:

$$\begin{aligned} G &\propto A_\omega B^2 f^{-2} \\ c &\propto A_\omega^{1/2} B^{1/2} f^{-1} \\ \hbar &\propto A_\omega^{1/2} B^{-1/2} f \end{aligned} \quad (90)$$

$\Lambda \propto f^{-2}$ and $m_i \propto f$, and if we impose $\omega = -1/3$ in this case we obtain:

$$\begin{aligned} G &= \text{const.} \\ c &= \text{const.} \\ \hbar &\propto A_\omega^{1/2} B^{-1/2} f^2 \end{aligned} \quad (91)$$

while $\Lambda \propto f^{-2}$ and $m_i \propto f$.

If the constants G, c, \hbar and Λ vary, then the Wheeler-DeWitt equation

$$\frac{d^2\psi}{df^2} - \frac{9\pi^2 f^2}{4l_p^4} \left(k - \frac{\Lambda f^2}{3} - \frac{8\pi G}{3c^4} A_\omega f^{-1-3\omega} \right) \psi = 0$$

it yields:

$$\left[\frac{d^2}{df^2} - \frac{9\pi^2}{4f^2} \left(k - \frac{d_0}{3} - \frac{8\pi d_\omega}{3} \right) \right] \psi = 0 \quad (92)$$

where the potential is defined by:

$$V(f) = \frac{9\pi^2}{4f^2} \left(k - \frac{d_0}{3} - \frac{8\pi d_\omega}{3} \right) \quad (93)$$

and where $d_\omega \in \mathbb{R}$ since

$$\begin{aligned} l_p &\propto f \\ \Lambda &\propto f^{-2} \\ G/c^2 &= B \\ c^2 &\propto A_\omega B f^{-1-3\omega} \end{aligned} \quad (94)$$

while the Schrödinger equation it yields:

$$A_\omega f^{2-3\omega} \frac{d^2\psi}{df^2} - \frac{9\pi^2}{4} A_\omega f^{-3\omega} \left(k - \frac{d_0}{3} - \frac{8\pi d_\omega}{3} \right) \psi = 0 \quad (95)$$

where the potential is defined by:

$$U(f) = \frac{9\pi^2}{4} A_\omega f^{-3\omega} \left(k - \frac{d_0}{3} - \frac{8\pi d_\omega}{3} \right) \quad (96)$$

if we simplify eq. (95) then it reduces to:

$$\frac{d^2\psi}{df^2} - \frac{9\pi^2}{4f^2} \left(k - \frac{d_0}{3} - \frac{8\pi d_\omega}{3} \right) \psi = 0 \quad (97)$$

it is observed that in this case both equations (92) and (97) are identical.

For example in the case of $\omega = 1/3$ equation (95) it yields:

$$A_\omega f \frac{d^2\psi}{df^2} - \frac{9\pi^2}{4} A_\omega f^{-1} \left(k - \frac{d_0}{3} - \frac{8\pi d_\omega}{3} \right) \psi = 0 \quad (98)$$

and if we use $\omega = -1/3$ it yields:

$$A_\omega f^3 \frac{d^2\psi}{df^2} - \frac{9\pi^2}{4} A_\omega f \left(k - \frac{d_0}{3} - \frac{8\pi d_\omega}{3} \right) \psi = 0 \quad (99)$$

Note that the value of the numerical constants $d_i \in \mathbb{R}$ are fundamental.

We would like to emphasize that in this paper all the constants are considered as variable, including the Planck's constant while in previous references it is considered only the variation of the constant Λ or the variation of G, c and Λ but no the Planck's constant \hbar ^{8,9}

4.1. Quantum tunneling.

The tunneling probability follows from the expression:

$$P \simeq \exp \left[-\frac{2}{\hbar} \int_0^{f_0} \sqrt{2m_p U(f')} df' \right] \quad (100)$$

where $U(f')$ follows from expression (96), since until now all the proposals only consider the potential $V(f)$ (93) despising all the “constants”. This expression is obviously dimensionally homogeneous while the usual ones are not.

Our potential is defined by:

$$U(f) = \frac{9\pi^2}{4} A_\omega f^{-3\omega} \left(k - \frac{d_0}{3} - \frac{8\pi d_\omega}{3} \right) \quad (101)$$

as we have seen above the “constants” m_p and \hbar vary as:

$$m_p \propto B^{-1} f \quad \hbar \propto A_\omega^{1/2} B^{-1/2} f^{3(1-\omega)/2} \quad (102)$$

therefore this probability may approximate by:

$$P \propto \exp \left[-\frac{2\sqrt{\tilde{\chi}}}{3-3\omega} \right] \quad (103)$$

where $\tilde{\chi} = 6\pi^2 (3k - d_0 - 8\pi d_\omega)$, we see that such probability only depends on the considered equation of state. If for example we take $\omega = 1$ (ultrastiff matter) then the probability vanished. This formula may help us to find an equation of state in a pre-big-bang scenario.

5. Conclusions

In this work we have try to show how all the constants vary when the conservation principle is taken into account. We have started revising previous results and then we have gone next to incorporate in our equations the condition $\text{div}T = 0$. This condition allows us to take into account mechanics of adiabatic matter creation in such a way that we have been able to get rid of the entropy problem. We have show how all the “constants” vary as well as all the physical quantities in the model in function of the equation of state and trying to generalize our results we have contemplated the possibility of considering a multi-fluid frameworks since these possibilities are present in the Wheeler-DeWitt equation. To study the variation of “constant” \hbar , we need to introduce it into the equations by imposing a special equation of state and to make assumptions about its behaviour but with the hypothesis of the conservation principle i.e. taking into account the condition $\text{div}T = 0$, we have seen that no previous hypotheses are needed. To validate all these results we have considered the W-DW equation.

We have studied from a dimensional point of view the Wheeler-DeWitt equation in the minisuperspace approximation. We have shown that the set of governing parameters is formed by $\varphi = (f, G, c, \hbar, \Lambda, A_\omega)$ (depending in each case) where f is

the only fundamental quantity and the set of unavoidable constants \mathcal{C} is constituted by $\mathcal{C} = \{G, c, \hbar, \Lambda, A_\omega\}$. A remarkable and surprising feature of the theory is the fact that it is independent of time, the only fundamental quantity is the scale factor f . As we have shown the multiplicity of the dimensional base is usually 3 and that a possible base could be formed by the Planck's system of units i.e. $B = \{G, c, \hbar, k_B\}$ showing in this way that is precisely in this framework where this system acquires a complete sense. Writing the Wheeler-DeWitt equation in a dimensionless way we have seen that the behaviour of the constants is the same that the obtained ones in the case of the standard cosmology. In this way we have deduced the Wheeler-DeWitt equation with all the constants varying as well as the Schrödinger equation (its reduced form). We have also seen that our model may arise via quantum tunneling.

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